

# ON THE SCHRÖDINGER-MAXWELL SYSTEM INVOLVING SUBLINEAR TERMS

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## Abstract

In this paper we study the coupled Schrödinger-Maxwell system

$$\begin{cases} -\Delta u + u + \phi u = \lambda \alpha(x) f(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where  $\alpha \in L^\infty(\mathbb{R}^3) \cap L^{6/(5-q)}(\mathbb{R}^3)$  for some  $q \in (0, 1)$ , and the continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is superlinear at zero and sublinear at infinity, e.g.,  $f(s) = \min(|s|^r, |s|^p)$  with  $0 < r < 1 < p$ . Depending on the range of  $\lambda > 0$ , non-existence and multiplicity results are obtained.

*Keywords:* Schrödinger-Maxwell system, sublinearity, non-existence, multiplicity.

## 1 Introduction

The problem of coupled Schrödinger-Maxwell equations

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + \omega u + e\phi u = g(x, u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = 4\pi e u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (SM)$$

has been widely studied in the recent years, describing the interaction of a charged particle with a given electrostatic field. The quantities  $m$ ,  $e$ ,  $\omega$  and  $\hbar$  are the mass, the charge, the phase, and the Planck's constant, respectively. The unknown terms  $u : \mathbb{R}^3 \rightarrow \mathbb{R}$  and  $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$  are the fields associated to the particle and the electric potential, respectively, while the nonlinear term  $g : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$  describes the interaction between the particles or an external nonlinear perturbation of the 'linearly' charged fields in the presence of the electrostatic field.

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System  $(SM)$  is well-understood for the model nonlinearity  $g(x, s) = \alpha(x)|s|^{p-1}s$  where  $p > 0$ ,  $\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}$  is measurable; various existence and multiplicity results are available for  $(SM)$  in the case  $1 < p < 5$ , see Azzollini and Pomponio [2], Benci and Fortunato [3, 4], Cerami and Vaira [7], Coclite [8], Coclite and Georgiev [9], D'Aprile and Wei [11, 12], D'Avenia [13], Kikuchi [15], and D'Avenia, Pisani and Siciliano [14] (for bounded domains). Via a Pohožaev-type argument, D'Aprile and Mugnai [10] proved the non-existence of the solutions  $(u, \phi)$  in  $(SM)$  for every  $p \in (0, 1] \cup [5, \infty)$  when  $\alpha = 1$ .

Besides of the model nonlinearity  $g(x, s) = \alpha(x)|s|^{p-1}s$ , important contributions can be found in the theory of the Schrödinger-Maxwell system when the right-hand side nonlinearity is more general, verifying various growth assumptions near the origin and at infinity. We recall two such classes of nonlinearities (for simplicity, we consider only the autonomous case  $g = g(x, \cdot)$ ):

(AR)  $g \in C(\mathbb{R}, \mathbb{R})$  verifies the global *Ambrosetti-Rabinowitz growth assumption*, i.e., there exists  $\mu > 2$  such that

$$0 < \mu G(s) \leq sg(s) \text{ for all } s \in \mathbb{R} \setminus \{0\}, \quad (1.1)$$

where  $G(s) = \int_0^s g(t)dt$ . Note that (1.1) implies the superlinearity at infinity of  $g$ , i.e., there exist  $c, s_0 > 0$  such that  $|g(s)| \geq c|s|^{\mu-1}$  for all  $|s| \geq s_0$ . Up to some further technicalities, by standard Mountain Pass arguments one can prove that  $(SM)$  has at least a nontrivial solution  $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ ; see Benci and Fortunato [4] for the pure-power case  $g(s) = |s|^{p-1}s$ ,  $3 < p < 5$ .

(BL)  $g \in C(\mathbb{R}, \mathbb{R})$  verifies the *Berestycki-Lions growth assumptions*, i.e.,

- $-\infty \leq \limsup_{s \rightarrow \infty} \frac{g(s)}{s^5} \leq 0$ ;
- $-\infty < \liminf_{s \rightarrow 0^+} \frac{g(s)}{s} \leq \limsup_{s \rightarrow 0^+} \frac{g(s)}{s} = -m < 0$ ;
- There exists  $s_0 \in \mathbb{R}$  such that  $G(s_0) > 0$ .

In the case when  $\omega = 0$  and  $e$  is small enough, Azzollini, D'Avenia and Pomponio [1] proved the existence of at least a nontrivial solution  $(u_e, \phi_e) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$  for the system  $(SM)$  via suitable truncation and monotonicity arguments.

The purpose of the present paper is to describe a new phenomenon for Schrödinger-Maxwell systems (rescaling the mass, the phase and the Planck's constant as  $2m = \omega = \hbar = 1$ ), by considering the non-autonomous eigenvalue problem

$$\begin{cases} -\Delta u + u + e\phi u = \lambda \alpha(x)f(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = 4\pi e u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (SM_\lambda)$$

where  $\lambda > 0$  is a parameter,  $\alpha \in L^\infty(\mathbb{R}^3)$ , and the continuous nonlinearity  $f : \mathbb{R} \rightarrow \mathbb{R}$  verifies the assumptions

$$(\mathbf{f1}) \quad \lim_{|s| \rightarrow \infty} \frac{f(s)}{s} = 0;$$

$$(\mathbf{f2}) \quad \lim_{s \rightarrow 0} \frac{f(s)}{s} = 0;$$

$$(\mathbf{f3}) \quad \text{There exists } s_0 \in \mathbb{R} \text{ such that } F(s_0) > 0.$$

**Remark 1.1** (a) Property **(f1)** is a *sublinearity growth assumption at infinity* on  $f$  which complements the Ambrosetti-Rabinowitz-type assumption (1.1).

(b) If **(f1)**-(**f3**) hold for  $f$ , then the function  $g(s) = -s + f(s)$  verifies all the assumptions in  $(BL)$  whenever  $1 < \max_{s \neq 0} \frac{2F(s)}{s^2}$ . Consequently, the results of Ambrosetti, Rabinowitz and Pomponio [1] can be applied also for  $(SM_\lambda)$ , guaranteeing the existence of at least one nontrivial pair of solutions when  $\lambda = \alpha(x) = 1$ , and  $e > 0$  is sufficiently small.

On account of Remark 1.1 (b), we could expect a much stronger conclusion when **(f1)**-(**f3**) hold. Indeed, the real effect of the sublinear nonlinear term  $f : \mathbb{R} \rightarrow \mathbb{R}$  will be reflected in the following two results.

Let  $e > 0$  be arbitrarily fixed. According to hypotheses **(f1)**-(**f3**), one can define the number

$$c_f = \max_{s \neq 0} \frac{|f(s)|}{|s| + 4\sqrt{\pi e} s^2} > 0. \quad (1.2)$$

We first prove a non-existence result for the system  $(SM_\lambda)$  whenever  $\lambda > 0$  is small enough. Namely, we have

**Theorem 1.1** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function which satisfies **(f1)**-(**f3**), and  $\alpha \in L^\infty(\mathbb{R}^3)$ . Then for every  $\lambda \in [0, \|\alpha\|_\infty^{-1} c_f^{-1})$  (with convention  $1/0 = +\infty$ ), problem  $(SM_\lambda)$  has only the solution  $(u, \phi) = (0, 0)$ .*

In spite of the above non-existence result, the situation changes significantly for larger values of  $\lambda > 0$ . Our main theorem reads as follows.

**Theorem 1.2** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function which satisfies **(f1)**-(**f3**), and  $\alpha \in L^\infty(\mathbb{R}^3) \cap L^{6/(5-q)}(\mathbb{R}^3)$  be a non-negative, non-zero, radially symmetric function for some  $q \in (0, 1)$ . Then there exist an open interval  $\Lambda \subset (\|\alpha\|_\infty^{-1} c_f^{-1}, \infty)$  and a real number  $\nu > 0$  such that for every  $\lambda \in \Lambda$  problem  $(SM_\lambda)$  has at least two distinct, radially symmetric, nontrivial pair of solutions  $(u_\lambda^i, \phi_\lambda^i) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ ,  $i \in \{1, 2\}$ , such that*

$$\|u_\lambda^i\|_{H^1} \leq \nu \text{ and } \|\phi_\lambda^i\|_{\mathcal{D}^{1,2}} \leq \nu. \quad (1.3)$$

**Remark 1.2** A Strauss-type argument shows that the solutions in Theorem 1.2 are homoclinic, i.e., for every  $\lambda \in \Lambda$  and  $i \in \{1, 2\}$ , we have

$$u_\lambda^i(x) \rightarrow 0 \text{ and } \phi_\lambda^i(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty.$$

**Example 1.1** Typical nonlinearities which fulfil hypotheses **(f1)**–**(f3)** are:

- (a)  $f(s) = \min(|s|^r, |s|^p)$  with  $0 < r < 1 < p$ .
- (b)  $f(s) = \min(s_+^r, s_+^p)$  with  $0 < r < 1 < p$ , where  $s_+ = \max(0, s)$ ;
- (c)  $f(s) = \ln(1 + s^2)$ .

The proof of Theorem 1.1 is based on a direct calculation. Theorem 1.2 is proved by means of a three critical point result of Bonanno [6] which is a refinement of a general principle of Ricceri [16, 17]. In Section 3 we give additional information concerning the location of the interval  $\Lambda$  which appears in Theorem 1.2.

*Notations and embeddings.*

- For every  $p \in [1, \infty]$ ,  $\|\cdot\|_p$  denotes the usual norm of the Lebesgue space  $L^p(\mathbb{R}^3)$ .
- The standard Sobolev space  $H^1(\mathbb{R}^3)$  is endowed with the norm  $\|u\|_{H^1} = (\int_{\mathbb{R}^3} |\nabla u|^2 + u^2)^{1/2}$ . Note that the embedding  $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$  is continuous for every  $p \in [2, 6]$ ; let  $s_p > 0$  be the best Sobolev constant in the above embedding.  $H_{\text{rad}}^1(\mathbb{R}^3)$  denotes the radially symmetric functions of  $H^1(\mathbb{R}^3)$ . The embedding  $H_{\text{rad}}^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$  is compact for every  $p \in (2, 6)$ .
- The space  $\mathcal{D}^{1,2}(\mathbb{R}^3)$  is the completion of  $C_0^\infty(\mathbb{R}^3)$  with respect to the norm  $\|\phi\|_{\mathcal{D}^{1,2}} = (\int_{\mathbb{R}^3} |\nabla \phi|^2)^{1/2}$ . Note that the embedding  $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$  is continuous; let  $d^* > 0$  be the best constant in this embedding.  $\mathcal{D}_{\text{rad}}^{1,2}(\mathbb{R}^3)$  denotes the radially symmetric functions of  $\mathcal{D}^{1,2}(\mathbb{R}^3)$ .

## 2 Preliminaries

Let  $e > 0$  be fixed. By the Lax-Milgram theorem it follows that for every  $u \in H^1(\mathbb{R}^3)$ , the equation

$$-\Delta \phi = 4\pi e u^2 \text{ in } \mathbb{R}^3, \quad (2.1)$$

has a unique solution  $\Phi[u] = \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ . Moreover, straightforward adaptation of [1, Lemma 2.1] and [18, Lemma 2.1] give the following basic properties of  $\phi_u$ .

**Proposition 2.1** *The map  $u \mapsto \phi_u$  has the following properties:*

- (a)  $\|\phi_u\|_{\mathcal{D}^{1,2}}^2 = 4\pi e \int_{\mathbb{R}^3} \phi_u u^2$  and  $\phi_u \geq 0$ ;
- (b)  $\|\phi_u\|_{\mathcal{D}^{1,2}} \leq 4\pi e d^* \|u\|_{12/5}^2$  and  $\int_{\mathbb{R}^3} \phi_u u^2 \leq 4\pi e d^{*2} \|u\|_{12/5}^4$ ;
- (c) *If the sequence  $\{u_n\} \subset H_{\text{rad}}^1(\mathbb{R}^3)$  weakly converges to  $u \in H_{\text{rad}}^1(\mathbb{R}^3)$  then  $\int_{\mathbb{R}^3} \phi_{u_n} u_n^2$  converges to  $\int_{\mathbb{R}^3} \phi_u u^2$ .*

We are interested in the existence of weak solutions  $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$  for the system  $(SM_\lambda)$ , i.e.,

$$\int_{\mathbb{R}^3} (\nabla u \nabla v + uv + e\phi uv) = \lambda \int_{\mathbb{R}^3} \alpha(x) f(u) v \text{ for all } v \in H^1(\mathbb{R}^3), \quad (2.2)$$

$$\int_{\mathbb{R}^3} \nabla \phi \nabla \psi = 4\pi e \int_{\mathbb{R}^3} u^2 \psi \text{ for all } \psi \in \mathcal{D}^{1,2}(\mathbb{R}^3), \quad (2.3)$$

whenever **(f1)** – **(f3)** hold and  $\alpha \in L^\infty(\mathbb{R}^3)$ . Note that all terms in (2.2)-(2.3) are finite; we will check only the right hand sides in both expressions, the rest being straightforward. First, **(f1)** and **(f2)** imply in particular that one can find a number  $n_f > 0$  such that  $|f(s)| \leq n_f |s|$  for all  $s \in \mathbb{R}$ . Thus, the right hand side of (2.2) is well-defined. Moreover, for every  $(u, \psi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$  we have

$$\begin{aligned} \int_{\mathbb{R}^3} u^2 |\psi| &\leq \left( \int_{\mathbb{R}^3} |u|^{12/5} \right)^{5/6} \left( \int_{\mathbb{R}^3} \psi^6 \right)^{1/6} \\ &= \|u\|_{12/5}^2 \|\psi\|_6 \\ &\leq s_{12/5}^2 d^* \|u\|_{H^1}^2 \|\psi\|_{\mathcal{D}^{1,2}} < \infty. \end{aligned}$$

For every  $\lambda > 0$ , we define the functional  $J_\lambda : H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \rightarrow \mathbb{R}$  by

$$J_\lambda(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^3} u^2 + \frac{e}{2} \int_{\mathbb{R}^3} \phi u^2 - \frac{1}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi|^2 - \lambda \mathcal{F}(u),$$

where

$$\mathcal{F}(u) = \int_{\mathbb{R}^3} \alpha(x) F(u).$$

It is clear that  $J_\lambda$  is well-defined and is of class  $C^1$  on  $H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ . Moreover, a simple calculation shows that its critical points are precisely the weak solutions for  $(SM_\lambda)$ , i.e., the relations

$$\left\langle \frac{\partial J_\lambda}{\partial u}(u, \phi), v \right\rangle = 0 \text{ and } \left\langle \frac{\partial J_\lambda}{\partial \phi}(u, \phi), \psi \right\rangle = 0,$$

give (2.2) and (2.3), respectively. Consequently, to prove existence of solutions  $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$  for the system  $(SM_\lambda)$ , it is enough to seek critical points of the functional  $J_\lambda$ .

Note that  $J_\lambda$  is a strongly indefinite functional; thus, the location of its critical points is a challenging problem in itself. However, the standard trick is to introduce a 'one-variable' energy functional instead of  $J_\lambda$  via the map  $u \mapsto \phi_u$ , see relation (2.1). More precisely, we define the functional  $I_\lambda : H^1(\mathbb{R}^3) \rightarrow \mathbb{R}$  by

$$I_\lambda(u) = J_\lambda(u, \phi_u).$$

On account of Proposition 2.1 (a), we have

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^3} u^2 + \frac{e}{4} \int_{\mathbb{R}^3} \phi_u u^2 - \lambda \mathcal{F}(u), \quad (2.4)$$

which is of class  $C^1$  on  $H^1(\mathbb{R}^3)$ . By using standard variational arguments for functionals of two variables, we can state the following result.

**Proposition 2.2** *A pair  $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$  is a critical point of  $J_\lambda$  if and only if  $u$  is a critical point of  $I_\lambda$  and  $\phi = \Phi[u] = \phi_u$ .*

Furthermore, since the equation (2.1) is solved throughout the relation (2.3), we clearly have that  $\frac{\partial J_\lambda}{\partial \phi}(u, \phi_u) = 0$ . Thus, the derivative of  $I_\lambda$  is given by

$$\begin{aligned} \langle I'_\lambda(u), v \rangle &= \left\langle \frac{\partial J_\lambda}{\partial u}(u, \phi_u), v \right\rangle + \left\langle \frac{\partial J_\lambda}{\partial \phi}(u, \phi_u) \circ \phi'_u, v \right\rangle \\ &= \left\langle \frac{\partial J_\lambda}{\partial u}(u, \phi_u), v \right\rangle \\ &= \int_{\mathbb{R}^3} (\nabla u \nabla v + uv + e\phi_u uv) - \lambda \int_{\mathbb{R}^3} \alpha(x) f(u) v. \end{aligned} \quad (2.5)$$

We conclude this section by recalling the following Ricceri-type three critical point theorem which plays a crucial role in the proof of Theorem 1.2 together with the principle of symmetric criticality restricting the functional  $I_\lambda$  to the space  $H_{\text{rad}}^1(\mathbb{R}^3)$ .

**Theorem 2.1** [6, Theorem 2.1] *Let  $X$  be a separable and reflexive real Banach space, and let  $E_1, E_2 : X \rightarrow \mathbb{R}$  be two continuously Gâteaux differentiable functionals. Assume that there exists  $u_0 \in X$  such that  $E_1(u_0) = E_2(u_0) = 0$  and  $E_1(u) \geq 0$  for every  $u \in X$  and that there exist  $u_1 \in X$  and  $\rho > 0$  such that*

- (i)  $\rho < E_1(u_1)$ ;
- (ii)  $\sup_{E_1(u) < \rho} E_2(u) < \rho \frac{E_2(u_1)}{E_1(u_1)}$ .

Further, put

$$\bar{a} = \frac{\zeta \rho}{\rho \frac{E_2(u_1)}{E_1(u_1)} - \sup_{E_1(u) < \rho} E_2(u)},$$

with  $\zeta > 1$ , assume that the functional  $E_1 - \lambda E_2$  is sequentially weakly lower semi-continuous, coercive and satisfies the Palais-Smale condition for every  $\lambda \in [0, \bar{a}]$ .

Then there is an open interval  $\Lambda \subset [0, \bar{a}]$  and a number  $\kappa > 0$  such that for each  $\lambda \in \Lambda$ , the equation  $E'_1(u) - \lambda E'_2(u) = 0$  admits at least three solutions in  $X$  having norm less than  $\kappa$ .

### 3 Proofs

**Proof of Theorem 1.1.** Let us fix  $0 \leq \lambda < \|\alpha\|_\infty^{-1} c_f^{-1}$  (when  $\alpha = 0$ , we choose simply  $\lambda \geq 0$ ), and assume that  $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$  is a solution for  $(SM_\lambda)$ . By choosing  $v := u$  and  $\psi := \phi$  in relations (2.2) and (2.3), respectively, we obtain that

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2 + e\phi u^2) = \lambda \int_{\mathbb{R}^3} \alpha(x) f(u) u,$$

and

$$\int_{\mathbb{R}^3} |\nabla \phi|^2 = 4\pi e \int_{\mathbb{R}^3} \phi u^2. \quad (3.1)$$

Moreover, choose also  $\psi := |u| \in \mathcal{D}^{1,2}(\mathbb{R}^3)$  in (2.3); we obtain that

$$4\pi e \int_{\mathbb{R}^3} |u|^3 = \int_{\mathbb{R}^3} \nabla \phi \nabla |u|,$$

thus,

$$4\sqrt{\pi} e \int_{\mathbb{R}^3} |u|^3 = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^3} \nabla \phi \nabla |u| \leq \int_{\mathbb{R}^3} \left( \frac{1}{4\pi} |\nabla \phi|^2 + |\nabla u|^2 \right).$$

Combining the above three relations and the definition of  $c_f$  from (1.2), this yields

$$\begin{aligned} \int_{\mathbb{R}^3} (u^2 + 4\sqrt{\pi} e |u|^3) &\leq \int_{\mathbb{R}^3} \left( |\nabla u|^2 + u^2 + \frac{1}{4\pi} |\nabla \phi|^2 \right) \\ &= \lambda \int_{\mathbb{R}^3} \alpha(x) f(u) u \\ &\leq \lambda \int_{\mathbb{R}^3} |\alpha(x)| |f(u)| |u| \\ &\leq \lambda \|\alpha\|_\infty c_f \int_{\mathbb{R}^3} (u^2 + 4\sqrt{\pi} e |u|^3). \end{aligned}$$

If  $\alpha = 0$ , then  $u = 0$ . If  $\alpha \neq 0$ , and  $0 \leq \lambda < \|\alpha\|_\infty^{-1} c_f^{-1}$ , the last estimates give that  $u = 0$ . Moreover, (3.1) implies that  $\phi = 0$  as well, which concludes the proof.  $\square$

**Remark 3.1** (a) The last estimates in the proof of Theorem 1.1 show that if  $f$  is a globally Lipschitz function with Lipschitz constant  $L_f > 0$  and  $f(0) = 0$ , then  $(SM_\lambda)$  has only the solution  $(u, \phi) = (0, 0)$  for every  $0 \leq \lambda < \|\alpha\|_\infty^{-1} L_f^{-1}$ , no matter if the assumptions **(f1)**-**(f3)** hold or not. In addition, if  $f$  fulfills **(f1)**-**(f3)** then  $c_f \leq L_f$ , and as expected, the range of those values of  $\lambda$ 's where non-existence occurs for  $(SM_\lambda)$  is larger than in the previous statement.

(b) If  $f(s) = \min(s_+^r, s_+^p)$  with  $0 < r < 1 < p$ , then  $L_f = p$  and  $c_f = \max_{s \neq 0} \frac{\min(s_+^r, s_+^p)}{|s| + 4\sqrt{\pi} e s^2} \leq \max_{s > 0} \min(s^{r-1}, s^{p-1}) = 1$  for every  $e > 0$ .

(c) If  $f(s) = \ln(1+s^2)$ , then  $L_f = 1$  and  $c_f = \max_{s \neq 0} \frac{\ln(1+s^2)}{|s| + 4\sqrt{\pi} e s^2} \leq \max_{s \neq 0} \frac{\ln(1+s^2)}{|s|} \approx 0.804$  for every  $e > 0$ .

**Proof of Theorem 1.2.** In the rest of this section we assume that the assumptions of Theorem 1.2 are fulfilled. For every  $\lambda \geq 0$ , let  $\mathcal{R}_\lambda = I_\lambda|_{H_{\text{rad}}^1(\mathbb{R}^3)} : H_{\text{rad}}^1(\mathbb{R}^3) \rightarrow \mathbb{R}$  be the functional defined by

$$\mathcal{R}_\lambda(u) = E_1(u) - \lambda E_2(u),$$

where

$$E_1(u) = \frac{1}{2}\|u\|_{H^1}^2 + \frac{e}{4} \int_{\mathbb{R}^3} \phi_u u^2 \text{ and } E_2(u) = \mathcal{F}(u), \quad u \in H_{\text{rad}}^1(\mathbb{R}^3). \quad (3.2)$$

To complete the proof of Theorem 1.2, some lemmas need to be proven.

**Lemma 3.1** *For every  $\lambda \geq 0$ , the functional  $\mathcal{R}_\lambda$  is sequentially weakly lower semicontinuous on  $H_{\text{rad}}^1(\mathbb{R}^3)$ .*

*Proof.* First, on account of Brézis [5, Corollaire III.8] and Proposition 2.1 (c), the functional  $E_1$  is sequentially weakly lower semicontinuous on  $H_{\text{rad}}^1(\mathbb{R}^3)$ . Now, due to **(f1)** and **(f2)**, it follows in particular that for every  $\varepsilon > 0$ , there exists  $c_\varepsilon > 0$  such that

$$|f(s)| \leq \varepsilon|s| + c_\varepsilon s^2 \text{ for all } s \in \mathbb{R}. \quad (3.3)$$

We assume that there exists a sequence  $\{u_n\} \subset H_{\text{rad}}^1(\mathbb{R}^3)$  which weakly converges to an  $u \in H_{\text{rad}}^1(\mathbb{R}^3)$ , but for some  $\delta > 0$ , we have

$$|E_2(u_n) - E_2(u)| > \delta \text{ for all } n \in \mathbb{N}. \quad (3.4)$$

In particular, we may assume that  $\{u_n\}$  is bounded in  $H_{\text{rad}}^1(\mathbb{R}^3)$ , and  $\{u_n\}$  strongly converges to  $u$  in  $L^3(\mathbb{R}^3)$ . By the standard mean value theorem, (3.3) and Hölder inequality, we obtain that

$$\begin{aligned} |E_2(u_n) - E_2(u)| &\leq \int_{\mathbb{R}^3} \alpha(x) |F(u_n) - F(u)| \\ &\leq \|\alpha\|_\infty \int_{\mathbb{R}^3} (\varepsilon(|u_n| + |u|) + c_\varepsilon(u_n^2 + u^2)) |u_n - u| \\ &\leq \varepsilon \|\alpha\|_\infty (\|u_n\|_{H^1} + \|u\|_{H^1}) \|u_n - u\|_{H^1} \\ &\quad + c_\varepsilon \|\alpha\|_\infty (\|u_n\|_3^2 + \|u\|_3^2) \|u_n - u\|_3. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary small and  $u_n \rightarrow u$  strongly in  $L^3(\mathbb{R}^3)$ , the last expression tends to 0, which contradicts (3.4). Consequently,  $E_2$  is sequentially weakly continuous, which completes out proof.  $\square$

**Lemma 3.2** *For every  $\lambda \geq 0$ , the functional  $\mathcal{R}_\lambda$  is coercive and satisfies the Palais-Smale condition.*



*Proof.* According to **(f1)** and **(f2)**, for every  $\varepsilon > 0$ , there exists  $\delta_\varepsilon \in (0, 1)$  such that

$$|f(s)| < \varepsilon|s| \text{ for all } |s| \leq \delta_\varepsilon \text{ and } |s| \geq \delta_\varepsilon^{-1}.$$

Since  $f \in C(\mathbb{R}, \mathbb{R})$ , there also exists a number  $M_\varepsilon > 0$  such that

$$\frac{|f(s)|}{|s|^q} \leq M_\varepsilon \text{ for all } |s| \in [\delta_\varepsilon, \delta_\varepsilon^{-1}],$$

where  $q \in (0, 1)$  is from the hypothesis for  $\alpha \in L^{6/(5-q)}(\mathbb{R}^3)$ . Combining the above two relations, we obtain that

$$|f(s)| \leq \varepsilon|s| + M_\varepsilon|s|^q \text{ for all } s \in \mathbb{R}. \quad (3.5)$$

Now, let us fix  $\lambda \geq 0$  arbitrarily, and choose  $\varepsilon := \frac{1}{(1+\lambda)\|\alpha\|_\infty}$  in (3.5). Thus, due to Proposition 2.1 (a), relation (3.5) and Hölder inequality, for every  $u \in H_{\text{rad}}^1(\mathbb{R}^3)$  we have

$$\begin{aligned} \mathcal{R}_\lambda(u) &\geq \frac{1}{2}\|u\|_{H^1}^2 + \frac{e}{4} \int_{\mathbb{R}^3} \phi_u u^2 - \lambda \int_{\mathbb{R}^3} \alpha(x) |F(u(x))| dx \\ &\geq \frac{1}{2}\|u\|_{H^1}^2 - \lambda \int_{\mathbb{R}^3} \alpha(x) \left( \frac{\varepsilon}{2} u^2 + \frac{M_\varepsilon}{q+1} |u|^{q+1} \right) dx \\ &\geq \frac{1}{2}(1 - \lambda \varepsilon \|\alpha\|_\infty) \|u\|_{H^1}^2 - \lambda \frac{M_\varepsilon}{q+1} \|\alpha\|_{6/(5-q)} s_6^{q+1} \|u\|_{H^1}^{q+1}. \end{aligned}$$

Since  $q+1 < 2$ , and on account of the choice of  $\varepsilon > 0$ , we conclude that  $\mathcal{R}_\lambda(u) \rightarrow \infty$  as  $\|u\|_{H^1} \rightarrow \infty$ , i.e.,  $\mathcal{R}_\lambda$  is coercive.

Now, let  $\{u_n\}$  be a sequence in  $H_{\text{rad}}^1(\mathbb{R}^3)$  such that  $\{\mathcal{R}_\lambda(u_n)\}$  is bounded and  $\|\mathcal{R}'_\lambda(u_n)\|_{H^{-1}} \rightarrow 0$ . Since  $\mathcal{R}_\lambda$  is coercive, the sequence  $\{u_n\}$  is bounded in  $H_{\text{rad}}^1(\mathbb{R}^3)$ . Thus, up to a subsequence, we may suppose that  $u_n \rightarrow u$  weakly in  $H_{\text{rad}}^1(\mathbb{R}^3)$ , and  $u_n \rightarrow u$  strongly in  $L^3(\mathbb{R}^3)$  for some  $u \in H_{\text{rad}}^1(\mathbb{R}^3)$ , and in particular, we have that

$$\langle \mathcal{R}'_\lambda(u), u - u_n \rangle \rightarrow 0 \text{ and } \langle \mathcal{R}'_\lambda(u_n), u - u_n \rangle \rightarrow 0 \quad (3.6)$$

as  $n \rightarrow \infty$ . Moreover,  $\{\phi_{u_n} u_n\}$  is bounded in  $L^{3/2}(\mathbb{R}^3)$ . Indeed, due to Proposition 2.1 (b), one has that

$$\begin{aligned} \|\phi_{u_n} u_n\|_{3/2}^{3/2} &= \int_{\mathbb{R}^3} \phi_{u_n}^{3/2} |u_n|^{3/2} \\ &\leq \left( \int_{\mathbb{R}^3} \phi_{u_n}^6 \right)^{1/4} \left( \int_{\mathbb{R}^3} u_n^2 \right)^{3/4} \\ &= \|\phi_{u_n}\|_6^{3/2} \|u_n\|_2^{3/2} \\ &\leq d^{*3/2} \|\phi_{u_n}\|_{\mathcal{D}^{1,2}}^{3/2} \|u_n\|_{H^1}^{3/2} \\ &\leq (4\pi e)^{3/2} d^{*3} \|u_n\|_{12/5}^3 \|u_n\|_{H^1}^{3/2} \\ &\leq 8d^{*3} (\pi e)^{3/2} s_{12/5}^3 \|u_n\|_{H^1}^{9/2} < \infty. \end{aligned}$$

Due to (2.5), a simple calculation shows that

$$\begin{aligned}\|u_n - u\|_{H^1}^2 &= \langle \mathcal{R}'_\lambda(u), u - u_n \rangle + \langle \mathcal{R}'_\lambda(u_n), u - u_n \rangle \\ &\quad + \lambda \int_{\mathbb{R}^3} \alpha(x) [f(u_n) - f(u)](u_n - u) dx \\ &\quad + e \int_{\mathbb{R}^3} [\phi_{u_n} u_n - \phi_u u](u_n - u) dx.\end{aligned}$$

The first two terms tend to 0, see (3.6). By means of (3.3) one has

$$\begin{aligned}\int_{\mathbb{R}^3} \alpha(x) |f(u_n) - f(u)| |u_n - u| dx &\leq \varepsilon \|\alpha\|_\infty (\|u_n\|_{H^1} + \|u\|_{H^1}) \|u_n - u\|_{H^1} \\ &\quad + \|\alpha\|_\infty c_\varepsilon (\|u_n\|_3^2 + \|u\|_3^2) \|u_n - u\|_3.\end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary small and  $u_n \rightarrow u$  strongly in  $L^3(\mathbb{R}^3)$ , the last terms tend to 0 as  $n \rightarrow \infty$ . Moreover, we clearly have that

$$\int_{\mathbb{R}^3} |\phi_{u_n} u_n - \phi_u u| |u_n - u| dx \leq \|\phi_{u_n} u_n - \phi_u u\|_{3/2} \|u_n - u\|_3.$$

Since  $\{\phi_{u_n} u_n\}$  is bounded in  $L^{3/2}(\mathbb{R}^3)$  and  $u_n \rightarrow u$  strongly in  $L^3(\mathbb{R}^3)$ , the last term also tend to 0. From the above facts, we conclude  $\|u_n - u\|_{H^1} \rightarrow 0$  as  $n \rightarrow \infty$ .  $\square$

**Lemma 3.3**  $\lim_{\rho \rightarrow 0^+} \frac{\sup\{E_2(u) : E_1(u) < \rho\}}{\rho} = 0$ .

*Proof.* A similar argument as in (3.3) shows that for every  $\varepsilon > 0$  there exists  $c_\varepsilon > 0$  such that

$$|F(s)| \leq \frac{\varepsilon}{4(1 + \|\alpha\|_\infty)} s^2 + c_\varepsilon |s|^3 \quad \text{for all } s \in \mathbb{R}. \quad (3.7)$$

For  $\rho > 0$  define the sets

$$W_\rho^1 = \{u \in H_{\text{rad}}^1(\mathbb{R}^3) : E_1(u) < \rho\}; \quad W_\rho^2 = \{u \in H_{\text{rad}}^1(\mathbb{R}^3) : \|u\|_{H^1}^2 < 2\rho\}.$$

On account of Proposition 2.1 (a), it is clear that  $W_\rho^1 \subseteq W_\rho^2$ . Moreover, by using (3.7), for every  $u \in W_\rho^2$  we have

$$\begin{aligned}E_2(u) &\leq \int_{\mathbb{R}^3} \alpha(x) |F(u)| \\ &\leq \int_{\mathbb{R}^3} \alpha(x) \left[ \frac{\varepsilon}{4(1 + \|\alpha\|_\infty)} u^2 + c_\varepsilon |u|^3 \right] \\ &\leq \frac{\varepsilon}{2} \rho + c_\varepsilon s_3^3 \|\alpha\|_\infty (2\rho)^{3/2}.\end{aligned}$$

Thus, one can fix a number  $\rho_\varepsilon > 0$  such that for every  $0 < \rho < \rho_\varepsilon$ , we have

$$0 \leq \frac{\sup_{u \in W_\rho^1} E_2(u)}{\rho} \leq \frac{\sup_{u \in W_\rho^2} E_2(u)}{\rho} < \frac{\varepsilon}{2} + 3c_\varepsilon s_3^3 \|\alpha\|_\infty \rho^{1/2} < \varepsilon,$$

which completes the proof.  $\square$

For any  $0 \leq r_1 \leq r_2$ , let  $A[r_1, r_2] = \{x \in \mathbb{R}^3 : r_1 \leq |x| \leq r_2\}$  be the closed annulus (perhaps degenerate) with radii  $r_1$  and  $r_2$ .

By assumption, since  $\alpha \in L^\infty(\mathbb{R}^3)$  is a radially symmetric function with  $\alpha \geq 0$  and  $\alpha \not\equiv 0$ , there are real numbers  $R > r \geq 0$  and  $\alpha_0 > 0$  such that

$$\operatorname{ess\,inf}_{x \in A[r, R]} \alpha(x) \geq \alpha_0. \quad (3.8)$$

Let  $s_0 \in \mathbb{R}$  from **(f3)**. For a fixed element  $\sigma \in (0, 1)$ , define  $u_\sigma \in H_{\text{rad}}^1(\mathbb{R}^3)$  such that

- (a)  $\operatorname{supp} u_\sigma \subseteq A[(r - (1 - \sigma)(R - r))_+, R]$ ;
- (b)  $u_\sigma(x) = s_0$  for every  $x \in A[r, r + \sigma(R - r)]$ ;
- (c)  $\|u_\sigma\|_\infty \leq |s_0|$ .

A simple calculation shows that

$$\|u_\sigma\|_{H^1}^2 \geq \frac{4\pi s_0^2}{3} [(r + \sigma(R - r))^3 - r^3], \quad (3.9)$$

and

$$\begin{aligned} E_2(u_\sigma) &\geq \frac{4\pi}{3} [\alpha_0 F(s_0) ((r + \sigma(R - r))^3 - r^3) - \|\alpha\|_\infty \max_{|t| \leq |s_0|} |F(t)| \times \\ &\quad \times (r^3 - (r - (1 - \sigma)(R - r))_+^3 + R^3 - (r + \sigma(R - r))^3)] \\ &\stackrel{\text{not.}}{=} M(\alpha_0, s_0, \sigma, R, r). \end{aligned} \quad (3.10)$$

We observe that for  $\sigma$  close enough to 1, the right-hand sides of both inequalities become strictly positive; choose such a number  $\sigma_0 \in (0, 1)$ .

*Proof of Theorem 1.2 (concluded).* We apply Theorem 2.1, by choosing  $X = H_{\text{rad}}^1(\mathbb{R}^3)$ , as well as  $E_1$  and  $E_2$  from (3.2). Due to Proposition 2.1 (a), we have at once that  $E_1(u) \geq 0$  for every  $u \in H_{\text{rad}}^1(\mathbb{R}^3)$ .

Due to relation (3.9) and Lemma 3.3, we may choose  $\rho_0 > 0$  such that

$$\begin{aligned} \rho_0 &< \frac{1}{2} \|u_{\sigma_0}\|_{H^1}^2 + \frac{e}{4} \int_{\mathbb{R}^3} \phi_{u_{\sigma_0}} u_{\sigma_0}^2; \\ \frac{\sup\{E_2(u) : E_1(u) < \rho_0\}}{\rho_0} &< \frac{4M(\alpha_0, s_0, \sigma_0, R, r)}{2\|u_{\sigma_0}\|_{H^1}^2 + e \int_{\mathbb{R}^3} \phi_{u_{\sigma_0}} u_{\sigma_0}^2}. \end{aligned}$$

By choosing  $u_1 = u_{\sigma_0}$ , hypotheses (i) and (ii) of Theorem 2.1 are verified. Define

$$\bar{a} = \frac{1 + \rho_0}{\frac{E_2(u_{\sigma_0})}{E_1(u_{\sigma_0})} - \frac{\sup\{E_2(u) : E_1(u) < \rho_0\}}{\rho_0}}. \quad (3.11)$$

Taking into account Lemmas 3.1 and 3.2, and put  $u_0 = 0$ , all the assumptions of Theorem 2.1 are verified. Therefore, there exist an open interval  $\Lambda \subset [0, \bar{a}]$  and a number  $\kappa > 0$  such that for each  $\lambda \in \Lambda$ , the equation  $\mathcal{R}'_\lambda(u) \equiv E'_1(u) - \lambda E'_2(u) = 0$  admits at least three solutions  $u_\lambda^i \in H_{\text{rad}}^1(\mathbb{R}^3)$ ,  $i \in \{1, 2, 3\}$ , having  $H^1$ -norms less than  $\kappa$ .

A similar argument as in [4, p. 416] shows that

$$\phi_{gu} = g\phi_u \text{ for all } g \in O(3), u \in H^1(\mathbb{R}^3),$$

where the compact group  $O(3)$  acts linearly and isometrically on  $H^1(\mathbb{R}^3)$  in the standard way. Consequently, the functional  $I_\lambda$  from (2.4) is  $O(3)$ -invariant. Moreover, since

$$H_{\text{rad}}^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : gu = u \text{ for all } g \in O(3)\},$$

the principle of symmetric criticality of Palais implies that the critical points  $u_\lambda^i \in H_{\text{rad}}^1(\mathbb{R}^3)$  ( $i \in \{1, 2, 3\}$ ) of the functional  $\mathcal{R}_\lambda = I_\lambda|_{H_{\text{rad}}^1(\mathbb{R}^3)}$  are also critical points of  $I_\lambda$ . Now, by Proposition 2.2 it follows that  $(u_\lambda^i, \phi_\lambda^i) \in H_{\text{rad}}^1(\mathbb{R}^3) \times \mathcal{D}_{\text{rad}}^{1,2}(\mathbb{R}^3)$  are critical points of  $J_\lambda$ , thus weak solutions for the system  $(SM_\lambda)$ , where  $\phi_\lambda^i = \phi_{u_\lambda^i}$ .

The norm-estimates in relation (1.3) follow by Proposition 2.1 (a), choosing  $\nu = \max(\kappa, 4\pi e d^{*2} s_{12/5}^2 \kappa^2)$ .  $\square$

**Remark 3.2** It is important to provide information about the location of the interval  $\Lambda$  which appears in Theorem 1.2. This step can be done in terms of  $\alpha_0$ ,  $s_0$ ,  $\sigma_0$ ,  $R$  and  $r$ . Due to Lemma 3.3, one can assume that  $\rho_0 < 1$  and

$$\frac{\sup\{E_2(u) : E_1(u) < \rho_0\}}{\rho_0} < \frac{E_2(u_{\sigma_0})}{2E_1(u_{\sigma_0})}.$$

On account of (3.11), we obtain

$$\bar{a} < \frac{4E_1(u_{\sigma_0})}{E_2(u_{\sigma_0})}. \quad (3.12)$$

In order to avoid technicalities, we assume in the sequel that  $r = 0$  which slightly restricts our study, imposing that  $\alpha$  does not vanish near the origin, see (3.8). The truncation function  $u_{\sigma_0} \in H_{\text{rad}}^1(\mathbb{R}^3)$  defined by

$$u_{\sigma_0}(x) = \begin{cases} 0 & \text{if } |x| > R, \\ s_0 & \text{if } |x| \leq \sigma_0 R, \\ \frac{s_0}{R(1-\sigma_0)}(R - |x|) & \text{if } \sigma_0 R < |x| \leq R, \end{cases}$$

verifies the properties (a)-(c) from above. Moreover, from Proposition 2.1 (b), we have

$$E_1(u_{\sigma_0}) \leq \frac{t}{2} + \pi e d^{*2} s_{12/5}^4 t^2 \stackrel{\text{not.}}{=} N(s_0, \sigma_0, R),$$

where

$$t = \frac{4\pi}{3} R s_0^2 \left[ R^2 + \frac{1 + \sigma_0 + \sigma_0^2}{1 - \sigma_0} \right].$$

Thus, combining the above estimation with relations (3.12) and (3.10), we obtain

$$\Lambda \subset \left( \|\alpha\|_\infty^{-1} c_f^{-1}, \frac{4N(s_0, \sigma_0, R)}{M(\alpha_0, s_0, \sigma_0, R, 0)} \right).$$

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